

# Flavor and Horizontal Symmetries

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After a brief introduction to what are the basic flavor questions to be addressed, I introduce the underlying ideas of horizontal symmetries, with group  $G_H$ . For the purposes of specific model building, it is useful to classify models according to the scale at which  $G_H$  is broken. I consider the three cases: below  $m_t$ ; somewhat above  $m_t$ ; and at  $M_{GUT}$ . After a discussion of the shadow sector in the  $E_6 \times E'_8$  superstring, there is a summary.

## §1. Flavor Questions.

We may identify three issues:

(i) *Replication.*

Surely the most basic flavor question is why there exists the replication of the quarks: u,d; c,s; t,b and similarly for the leptons:  $\nu_e, e; \nu_\mu, \mu; \nu_\tau, \tau$ .

Why are there three families? Although there are papers about this topic, it will not be my subject here.

(ii) *Fermion mass hierarchy.*

If I define  $\lambda = \sin\theta_C \simeq 0.22$ , the Cabibbo angle, as a "small" parameter then for the up-type (Q=+2/3) quarks:

$$m_u/m_t \sim \lambda^8; \quad m_c/m_t \sim \lambda^4 \quad (1.1)$$

while for the down-type quarks (Q = -1/3):

$$m_d/m_b \sim \lambda^4; \quad m_s/m_b \sim \lambda^2. \quad (1.2)$$

The charged lepton masses approximately satisfy:

$$m_e/m_\tau \sim \lambda^4; \quad m_\mu/m_\tau \sim \lambda^2. \quad (1.3)$$

These masses are evaluated at the GUT scale  $\sim 2 \times 10^{16}$  GeV. Whence do such hierarchies arise?

(iii) *Mixing Hierarchy.*

When the quark mass matrices are diagonalized with the usual bi-unitary transformations:

$$U_L M^U U_R^\dagger = \text{diag}(m_u, m_c, m_t) \quad D_L M^D D_R^\dagger = \text{diag}(m_d, m_s, m_b) \quad (1.4)$$

and the CKM matrix is constructed by  $V_{CKM} = U_L D_L^\dagger$  one finds that its elements have the hierarchy:

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$$|V_{ii} \sim 1| > |V_{12}| > |V_{23}| > |V_{13}| \quad (1.5)$$

where these four elements are of order  $1, \lambda, \lambda^2, \lambda^3$  respectively. Whence do these hierarchies come? .....is the third and last flavor question.

Note that the fermion and mixing hierarchies speak to 12 of the 19 parameters of the Standard Model: the  $m_i$  and the  $\Theta_k$ . The others 7 are: the three couplings  $\alpha_i$ ; the two CP parameters  $\delta$  and  $\bar{\theta}$ ; and the two scales  $M_W$  and  $M_H$  arising in the electroweak symmetry breaking.

## §2. Horizontal Symmetries

Let the horizontal symmetry group be  $G_H$ . There are choices to be made about  $G_H$ : whether it is global or gauged, finite or infinite(Lie), abelian or non-abelian.

My choices will be: gauged, finite and non-abelian.

Finite abelian groups  $(Z_N)^1$  and finite non-abelian groups  $(S_N \text{ only})^2$  have been studied previously.

What I mean by a gauged finite group will be discussed below.

First I offer a brief review of all finite groups of order  $g \leq 31$ . [One usually stops at  $g = 2^n - 1$  because of the richness of  $g = 2^n$ ] In this range there are, according to standard textbooks<sup>4)</sup>, 93 inequivalent groups of which 48 are abelian and 45 are non-abelian.  $g \geq 32$  might be interesting too but lower  $g$  is surely more economic.

Let me deal quickly with all the abelian cases. The building block is  $Z_p$  where the elements are the  $p^{\text{th}}$  roots of unity,  $e^{2\pi i/p}$ . The only fact one needs is that  $Z_p \times Z_q$  is equivalent to  $Z_{pq}$  if and only if p,q have no common prime factor. So if I decompose the order  $g$  of the group into its prime factors  $g = \prod_i p_i^{k_i}$  then the number of inequivalent abelian finite groups at order  $g$  is  $N_a(g) = \prod_{k_i} P(k_i)$ , where  $P(\nu)$  is the number of ordered partitions of  $\nu$ . For example,  $P(1, 2, 3, 4) = 1, 2, 3, 5$ .

For the cases  $g \leq 31$ , one finds  $N_a(g) = 1$  *except* that  $N_a(g) = 2$  for  $g = 4, 9, 12, 18, 20, 25, 28$ ;  $N_a(g) = 3$  for  $18, 24, 27$ ; and  $N_a(16) = 5$ .

Thus, adding these results gives the required answer of 48 abelian groups with  $g \leq 31$ . These will not be considered further here.

Now, and for the remainder of the talk, I shall consider non-abelian finite groups.

The best known are the  $S_N$  permutation (or symmetric) groups with order  $g = N!$  Since  $g$  grows so rapidly, only  $N=3, 4$  are in our range. Generally,  $S_N \subset O(N-1)$  and can be interpreted geometrically as the symmetry of a regular  $N$ -plex in  $(N-1)$  spatial dimensions.

Next come the dihedral groups  $D_N$ , with order  $g = 2N$ , which are subgroups of  $O(3)$ . The geometrical interpretation is the symmetry of a 2-sided planar regular  $N$ -agon in 3 spatial dimensions where the polygon is treated as a "two-faced" entity.

There is the one  $g = 12$  tetrahedral group (T) which is the even-permutation subgroup of  $S_4$ .

Spinorial generalizations (doubles) of the  $D_N$  are the dicyclic groups  $Q_{2N}$  which have order  $g = 4N$  and are subgroups of  $SU(2)$  rather than  $O(3)$ .

The majority (32) of the non-abelian groups with order  $g \leq 31$  are made from

$S_N, D_N, T, Q_{2N}$  as follows:

$g = 6$ :  $D_3(\equiv S_3)$ .

$g = 8$ :  $D_4; Q \equiv Q_4$ .

$g = 10$ :  $D_5$ .

$g = 12$ :  $D_6; Q_6; T$ .

$g = 14$ :  $D_7$ .

$g = 16$ :  $D_8; Q_8; Z_2 \times D_4; Z_2 \times Q$ .

$g = 18$ :  $D_9; Z_3 \times D_3$ .

$g = 20$ :  $D_{10}; Q_{10}$ .

$g = 22$ :  $D_{11}$ .

$g = 24$ :  $D_{12}; Q_{12}; Z_2 \times D_6; Z_2 \times Q_6; Z_2 \times T; Z_3 \times D_4; D_3 \times Q; Z_4 \times D_3; S_4$ .

$g = 26$ :  $D_{13}$ .

$g = 28$ :  $D_{14}; Q_{14}$ .

$g = 30$ :  $D_{15}; D_5 \times Z_3; D_3 \times Z_5$ . There are another 13 which are twisted products

of  $Z_n$ :

$g = 16$ :  $Z_2 \tilde{\times} Z_8(two); Z_4 \tilde{\times} Z_4; Z_2 \tilde{\times} (Z_2 \times Z_4)(two)$ .

$g = 18$ :  $Z_2 \tilde{\times} (Z_3 \times Z_3)$ .

$g = 20$ :  $Z_4 \tilde{\times} Z_5$ .

$g = 21$ :  $Z_3 \tilde{\times} Z_7$ .

$g = 24$ :  $Z_3 \tilde{\times} Z_q; Z_3 \tilde{\times} Z_8; Z_3 \tilde{\times} D_4$ .

$g = 27$ :  $Z_3 \tilde{\times} Z_9; Z_3 \tilde{\times} (Z_3 \times Z_3)$ .

Of these groups  $Q_{2N}$  is of most interest to model-building<sup>3), 4), 5), 6)</sup>.

The group  $Q_{2N}$  with order  $g = 4N$  has 4 singlet representations  $1, 1', 1'', 1'''$  and the  $(N - 1)$  doublets  $2_{(j)}, 1 \leq j \leq (N - 1)$ . The doublet multiplication is:

$$2_{(j)} \times 2_{(k)} = 2_{|j-k|} + 2_{\min(j+k, N-j-k)} \quad (2.1)$$

with the generalized notation:

$$2_{(0)} \equiv 1 + 1' \quad 2_{(N)} \equiv 1'' + 1''' \quad (2.2)$$

$Q_{2N}$  has only singlet and doublet representations.

To obtain a clearer intuitive understanding of  $Q_{2N}$ , it is defined by the algebra:

$$A^{2N} = E \quad B^2 = A^N \quad ABA = B$$

The elements are

$$E, A, A^2, A^3, \dots, A^{2N-1}, B, AB, A^2B, A^3B, \dots, A^{2N-1}B$$

A simple matrix representation is:

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with  $\theta = \pi/N$  because then

$$A = \begin{pmatrix} \cos N\theta & \sin N\theta \\ -\sin N\theta & \cos N\theta \end{pmatrix} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

and for B I use the simple matrix

$$B = \begin{pmatrix} -i & \\ & -i \end{pmatrix}$$

$Q_{2N}$  is thus the full  $SU(2)$  symmetry of a planar regular N-agon where rotation by  $2\pi$  gives a sign -1; only rotation by  $4\pi$  brings back the identity.

### 2.1. Gauging $G_H = Q_{2N}$

Gauging  $G_H$ , a finite group, is subtle as is immediately seen by considering the covariant derivative. In fact, at a local level it is meaningless in a flat spacetime neighborhood. Globally, with respect to topological aspects of the spacetime manifold, it is best done by gauging  $SU(2)_H \supset Q_{2N}$  then spontaneously breaking to  $Q_{2N}$ . If this is at a high scale, the effective theory has no gauge field - but consistency with wormholes is preserved.

Gauging leads to consistency requirements:

- (a) Chiral fermions must be in complete representations of  $SU(2)_H$ .
- (b)  $(SU(2)_H)^2 Y$  anomalies must cancel.
- (c) Witten's global  $SU(2)_H$  anomaly must cancel.

(b) and (c) are straightforward but (a) requires the dictionary for embedding  $Q_{2N} \subset SU(2)$ . This is actually a simple pattern:

$$SU(2) \rightarrow Q_{2N}$$

$$1 \rightarrow 1$$

$$2 \rightarrow 2_1$$

$$3 \rightarrow 1' + 2_2$$

$$4 \rightarrow 2_1 + 2_3$$

$$5 \rightarrow 1 + 2_2 + 2_4$$

and so on. The infinite sequence is clear from the above.

## §3. Model Building with Horizontal $Q_{2N}$ Symmetry

There are five "triples" in the standard model for which  $Q_{2N}$  assignments must be made:

- (1)  $(t, b)_L, (c, s)_L, (u, d)_L$ .
- (2)  $t_R, c_R, u_R$ .
- (3)  $b_R, s_R, d_R$ .
- (4)  $(\nu_\tau, \tau)_L, (\nu_\mu, \mu)_L, (\nu_e, e)_L$ .
- (5)  $\tau_R, \mu_R, e_R$ .

These must be assigned to anomaly-free complete representations of  $SU(2)_H \supset G_H \equiv Q_{2N}$ . The technical details depend on the  $G_H$  breaking scale. I shall consider: A 10 GeV, B 10 TeV, C  $10^{16}$  GeV.

3.1.  $\Lambda_H \sim 10\text{GeV}(> m_b)$ 

Assignments are arranged around the top quark such that:

t mass is a  $G_H$  singlet.

$b, \tau$  masses break  $G_H \rightarrow G'$ .

c mass breaks  $G' \rightarrow G''$

$s, \mu$  masses break  $G'' \rightarrow G'''$

At the same time, I exclude, or minimize, additional fermions and demand full anomaly cancellation.

It is possible to show<sup>4)</sup> that of all the 93 finite groups with  $g \leq 31$ , only the dicyclic groups remain as candidates for  $G_H$ . The simplest model uses  $Q_6$ . Recall that:

$$1 \rightarrow 1$$

$$2 \rightarrow 2_{(1)}$$

$$3 \rightarrow 1' + 2_{(2)}$$

so for the five triples the only possible assignments are:

$$1 + 1 = 1$$

$$1' + 2_{(2)}$$

$$1 + 2_{(1)}$$

For the  $Q_6$  model, the assignments are:

$$\left. \begin{pmatrix} t \\ b \end{pmatrix}_L \right\} 1 \quad \begin{pmatrix} t_R & 1 \\ b_R & 1' \end{pmatrix} \quad \left. \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \right\} 1 \quad \tau_R \quad 1'$$

$$\left. \begin{pmatrix} c \\ s \\ u \\ d \end{pmatrix}_L \right\} 2_S \quad \begin{pmatrix} c_R & 1 \\ u_R & 1 \\ s_R & \\ d_R & \end{pmatrix} \left. \right\} 2 \quad \left. \begin{pmatrix} \nu_\mu \\ \mu \\ \nu_e \\ e \end{pmatrix}_L \right\} 2_S \quad \left. \begin{pmatrix} \mu_R \\ e_R \end{pmatrix} \right\} 2$$

The mass matrix textures are:

$$U = \begin{pmatrix} \langle 2_S \rangle & \langle 2_S \rangle \\ \langle 1 \rangle & \langle 1 \rangle \end{pmatrix}$$

and:

$$D = L = \begin{pmatrix} \langle 1'' + 1''' + 2_S \rangle & \langle 2_S \rangle \\ \langle 2 \rangle & \langle 1' \rangle \end{pmatrix}$$

The symmetry-breaking steps involve a VEV to a  $Q_6$  singlet  $\Rightarrow t$  mass; then a VEV to a  $1'$  breaks  $Q_6$  to  $Z_6$ , providing  $b, \tau$  masses. A  $(1, 2_{(1)})$  VEV gives the  $c$  mass and finally a  $(2, 1'' + 1''')$  VEV gives the  $s, \mu$  masses.

Some remarks on the  $Q_6$  model:

(i) The  $(SU(2)_H)^2 Y$  anomaly cancellation requires certain extra singlets predicted to lie between 50GeV and 200GeV.

(ii) The hierarchy of Yukawa couplings has been removed: they now all lie between 0.1 and 1.0.

(iii) It provides a first step to understanding why the top quark mass is so different from the other quark masses.

### 3.2. Breaking of $G_H$ at $\geq 1TeV$

Here we shall use non-abelian horizontal symmetry in connection with derivation of an ansatz for the texture zeros in the quark mass matrices.

The horizontal symmetry will be again  $Q_{2N}$ , but now the Froggatt-Nielsen mechanism<sup>7)</sup> becomes an essential part of mass generation. This means that additional vector-like pairs of fermions, at high scale, are present - and these must likewise be assigned to representations of  $G_H$ .

The quark assignments are:

$$\left\{ \begin{array}{l} \left( \begin{array}{c} t \\ b \end{array} \right)_L \\ \left( \begin{array}{c} c \\ s \end{array} \right)_L \\ \left( \begin{array}{c} u \\ d \end{array} \right)_L \end{array} \right\} \quad \begin{array}{c} 2_{(2)} \\ 2_{(2)} \\ 1' \end{array} \quad \left\{ \begin{array}{l} t_R \\ c_R \\ b_R \\ s_R \\ u_R \\ d_R \end{array} \right\} \quad \begin{array}{c} 2_{(2)} \\ 2_{(1)} \\ 1' \\ 1 \end{array}$$

The lepton doublets are correspondingly  $(2_{(1)} + 1)$  and the singlets  $(2_{(2)} + 1')$ . This assignment is completely anomaly-free. To consider the quark masses I put the SM Higgs doublet in to down quarks. Their VEVs give masses only to the third family. The other elements of the up-quark mass matrix  $M_U$  arise from Froggatt-Nielsen graphs shown in Fig. 1.

There are second graphs, not shown in Fig.1, for  $(M_U)_{32}, (M_U)_{31}$ . Similar graphs for  $(M_D)$  lead to the textures:

$$M_U = \begin{pmatrix} 0 & 0 & \lambda^4 \\ 0 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix} \quad (3.1)$$

$$M_D = \begin{pmatrix} 0 & \lambda^4 & 0 \\ \lambda^4 & \lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.2)$$

Here, as before,  $\lambda \simeq \sin\theta_C$ . The expansion parameter is also identified with the ratios:

$$\lambda = \langle S_i \rangle / M_{\text{odd}} \quad \lambda^2 = \langle S_i \rangle / M_{\text{even}} \quad (3.3)$$

where the "odd"  $Q_6$  doublets occur in the  $SU(2)_H$  spinor representations ; the "even" in the vector ones.

Another symmetric texture (the only other attainable one with five zeros) is:

$$M_U = \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \quad (3.4)$$

$$M_D = \begin{pmatrix} 0 & \lambda^4 & 0 \\ \lambda^4 & \lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.5)$$

Three other phenomenologically-viable textures<sup>8)</sup> which are symmetric with five zeros are not attainable and hence disfavored.

The postulation of zeros in the mass matrices reduces the number of free parameters in the low energy theory. This now has a dual description in terms of a horizontal symmetry  $Q_{2N} \subset SU(2)_H$ .

This  $SU(2)_H$  could arise from a GUT group or directly from a superstring.

My main point is that derivation of the values of the masses in a putative theory of everything may likely involve a horizontal symmetry, probably gauged, as an important intermediate step. The two simple cases given above illustrate how this can happen.

### 3.3. $G_H$ Breaking $\geq M_{GUT}$

Again the Froggatt-Nielsen mechanism is used, invoking a spectrum of super-heavy fermions in vector-like pairs.

The symmetry group<sup>6)</sup> is  $SU(5) \times SU(5) \times SU(2)_H$ . The  $SU(2)_H$  is broken to  $Q_{12}$  at  $M(SU(2))$  and the  $SU(5) \times SU(5)$  is broken to a diagonal  $SU(5)$  at  $M_{GUT}$ . Supersymmetry is broken near the weak scale.

The light, heavy and superheavy fermion contents are in the following list of chiral  $SU(5) \times SU(5)$  supermultiplets. The third entry denotes the content under  $S(2)_H \rightarrow Q_{12}$ .

$$(10, 1, 1 \rightarrow 1(T))$$

$$(10, 1, 4 \rightarrow 2_1 + 2_3)$$

$$(10, 1, 7 \rightarrow 1' + 2_2 + 2_4 + 2_6)$$

$$(\overline{10}, 1, 4 \rightarrow 2_1 + 2_3)$$

$$(\overline{10}, 1, 7 \rightarrow 1' + 2_2 + 2_4 + 2_6)$$

$$(\bar{5}, 1, 3 \rightarrow 1' + 2_2)$$

$$(\bar{5}, 1, 6 \rightarrow 2_1 + 2_3 + 2_5(H_d/b))$$

$$(5, 1, 1(H_u))$$

$$(5, 1, 3 \rightarrow 1' + 2_2)$$

$$(5, 1, 4 \rightarrow 2_1 + 2_3)$$

$$(1, 10, 1)$$

$$(1, 10, 2 \rightarrow 2_1(Q))$$

$$(1, \bar{10}, 1)$$

$$(1, \bar{5}, 2 \rightarrow 2_1(D))$$

This list is seen to be relatively short when one realizes that it includes all the vector-like F-N superheavy fermions and the light chiral fermions.

The effective theory at the weak scale contains only the fermions of the standard model transforming as follows:  $(u, d)_L, (c, s)_L$  as  $Q(2_1)$ ;  $(t, b)_L$  as  $T(1)$ ;  $u_R, c_R$  as  $U(2_1)$ ;  $d_R, s_R$  as  $D(2_1)$ ;  $t_R$  as  $T(1)$ ;  $b_R$  as  $H_d/b(2_5)$ .

This gives supersymmetry without R-parity.  $b - \tau/H_d$  are in a horizontal doublet incompatible with the usual R-parity, but matter parity arises here from group properties of  $G_H$ . The " $\mu$  problem" ( $\bar{5}5$  term at tree level) is also solved here by  $G_H$ .

The mass matrix textures are:

$$M_U = \begin{pmatrix} \lambda^8 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.6)$$

$$M_D = \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^4 & \lambda^3 & \lambda^3 \\ 0 & \lambda^3 & 1 \end{pmatrix} \quad (3.7)$$

This is a phenomenologically viable SusyGUT for the low energy parameters.



### §4. Shadow $E'_8$ Sector

For the heterotic string, the  $E_8 \times E'_8$  becomes on Calabi-Yau compactification typically  $E_6 \times E'_8$  and the  $E_6$  can become  $SU(3)^3$ , for example. In the visible sector one then has unification at  $M_{GUT} \sim 2 \times 10^{16} GeV$  with  $\alpha_{GUT}^{-1} \sim 25$ .

The  $SU(3)^3$  couplings have  $\beta = 0$  and consistency dictates an  $M_{string} = 3.5 \times 10^{17} GeV$ . Bridging across to the shadow sector one can choose an  $SU(N)$  gauge subgroup of  $E'_8$  such that  $\alpha_N$  becomes  $O(1)$  at a scale where a gluino condensate may break supersymmetry. This suggests *e.g.*  $SU(5) \times SU(4) \times U(1)$ . It is possible that the shadow photino can act as cosmological dark matter<sup>9)</sup>.

### §5. Summary

The flavor questions for fermion mass and mixing hierarchies may require gauged horizontal symmetries and the above examples illustrate how the dicyclic groups are well-suited. The covering  $SU(2)_H$  is gauged for consistency. The cases I have described show how this can happen and how it can give phenomenologically acceptable mass matrices and mixings.

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